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Optimal packing of induced stars in a graph

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Abstract

We consider simple undirected graphs. An edge subset A of G is called an *induced n-star packing* of G if every component of the subgraph $G[A]$ induced by A is a star with at most n edges and is an induced subgraph of G. We consider the problem of finding an induced n-star packing of G that covers the maximum number of vertices. This problem is a natural generalization of the classical matching problem. We show that many classical results on matchings (such as the Tutte 1-Factor Theorem, the Berge Duality Theorem, the Gallai–Edmonds Structure Theorem, the Matching Matroid Theorem) can be extended to induced n-star packings in a graph.

1. Introduction

The problem of finding a maximum matching in a graph is well known. Many interesting results (that are classical now) concerning this problem have been found by Tutte, Edmonds, Berge, Gallai, Fulkerson, Lovasz and many others, see [14]. One of the ideas to generalize the matching theory is to try to replace the edges in a matching by some subgraphs of prescribed type. Let G be a graph, and $\mathscr{F}$ be a family of subgraphs of G. An edge subset A of G is called an *$\mathscr{F}$-packing of G* if every component of the subgraph $G[A]$ induced by A in G is a graph in the family $\mathscr{F}$. The *$\mathscr{F}$-packing problem* consists of finding an $\mathscr{F}$-packing A of G that covers the maximum number of vertices. If $\mathscr{F}$ consists of all 2-cliques (all subgraphs of one edge) of G then the $\mathscr{F}$-packing problem is precisely the classical maximum matching problem. The $\mathscr{F}$-packing problem has extensively been studied by many authors for different classes of families $\mathscr{F}$. Comprehensive surveys of results obtained so far can be found in [3,5,12].

Clearly the properties of the $\mathscr{F}$-packing problem essentially depend on the family $\mathscr{F}$. It is not surprising that the problem turns out to be NP-complete for most of the families $\mathscr{F}$ (e.g. [4,11,13]). Surprisingly the problem can be solved in polynomial time for some

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non-trivial classes of families $\mathscr{F}$, and many important results in matching theory can be generalized in those cases (e.g. [3,6,10,13]).

Let $\mathscr{F}$ be the set of stars of G with at least one and at most n edges. Then we have a *star packing problem* $\mathbf{S}_n$. This problem is one of the simplest generalizations of the matching problem that turns out to be 'good' [1,6,10] (this also follows easily from the approach developed in this paper). The nature of most of the known 'good' $\mathscr{F}(G)$-packing problems are similar to that for the star packing problem.

In this paper we consider a *star packing problem with an additional condition*. A new requirment is that every star in an $\mathscr{F}$-packing should be an *induced subgraph* of G. In other words we consider an $\mathscr{F}$-packing problem $\mathbf{IS}_n$ where $\mathscr{F} := \mathscr{F}_n(G)$ is the set of all induced stars of G having at most n edges.

Main results are described in Section 2. In Section 3 we introduce the notions of alternating and augmenting trails as well as passive paths and active trails, and describe some important properties of such trails. Analysis of induced star packings without augmenting trails is given in Section 4. Duality Theorem 2.3 (Section 5), the matroid results (Section 6), and Structure Theorem 2.6 (Section 7) on induced star packings are easy consequences of this analysis. In Section 8 we give a polynomial-time algorithm for solving the induced star packing problem in a graph. [1]

2. Main results

The notion and facts on graphs and matroids that are used but not described here can be found in [2,15], respectively. We consider undiracted graphs without loops or parallel edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively.

Given an edge subset A of G let $G[A]$ denote the *subgraph of G induced by A*, i.e. $E(G[A]) = A$ and $V(G[A])$ consists of the vertices of G covered by A (or incident to at least one edge in A).

Given a vertex subset B of G let $G[B]$ (or sometimes simply $\dot{B}$) denote the *subgraph of G induced by B*, i.e. $G[B] = G\setminus(V(G)\setminus B)$. An *induced subgraph of G* is a subgraph induced by a vertex subset of G. In other words a subgraph of G is *induced* if it can be obtained from G by deleting some vertices of G.

We write $Z[X]$ instead of $Z(G[X])$, for example, $V[X]$ instead of $V(G[X])$, $E[X]$ instead of $E(G[X])$, etc.

Given a subgraph H and an edge subset U of G, we put $U(H) = U \cap E(H)$.

A *star* S is a connected graph in which all the edges have a common vertex.

[1] The results of the paper were presented at RUTCOR Seminar, Rutgers University, in 1991, at the 23rd Southeastern International Conference on Combinatorics, Graph Theory and Computing in February 1992, at the Seymour–Schrijver Workshop on disjoint paths in DIMACS in November 1992, and at seminars in MIT, University of Waterloo, Simon Fraiser University, MacMaster University, University of Quebec in Montreal, and other institutions.

When considering the problem $\mathbf{IS}_n$ of packing induced stars in a graph, we need the following notion.

We say that an edge set A of G is an *induced n-star packing of G* (or simply an *induced n-packing of G*) if every component S of $G[A]$ is an induced subgraph of G isomorphic to a star with at most n edges.

An induced n-star packing A is called *perfect* if $V[A] = V(G)$, i.e. if A covers all the vertices of the graph.

Clearly if n is at least the maximum vertex degree of G then there are actually no restriction on the size of stars in an induced n-star packing, and so in this case an induced n-star packing of G will be called an *induced star packing*.

When considering the problem $\mathbf{S}_n$ of packing stars (not necessarily induced) in a graph, we will have the corresponding notions of an *n-star packing* (or simply *n-packing*), a *perfect n-star packing*, *star packing*, and *perfect star packing*.

Let $\mathscr{P}\mathscr{I}\mathscr{S}_n(G)$ denote the set of induced n-star packings of G.

The problem $\mathbf{IS}_n$ we are going to consider is to find an induced n-packing A in G that covers the maximum number of vertices, i.e. $|V[A]| = \max\{V[X]\colon X \in \mathscr{P}\mathscr{I}\mathscr{S}_n(G)\}$. Such an induced n-packing A in G is called *vertex maximum* or simply *V-maximum*.

In particular we want to characterize graphs having a perfect induced n-packing.

Clearly $\mathbf{IS}_1$ (as well as $\mathbf{S}_1$) is a matching problem. Unless otherwise is stated explicitly we assume throughout the paper that n is an integer at least 2.

A graph G is called IS_n*-critical* if G has no perfect induced n-packing but $G\backslash x$ has a perfect induced n-packing for every vertex x of G. An IS_1-critical graph is called usually *matching-critical* or *hypomatchable*.

Let $\mathrm{cr}_n(G)$ denote the number of IS_n-critical components of G. Let $W[A]$ denote the set of vertices of G that are not covered by A, i.e. $W[A] = V(G)\backslash V[A]$. It turns out that the following duality theorem holds.

Theorem 2.1. *Let n be a positive integer. Then*

$$\min\{|W[A]|\colon A \in \mathscr{P}\mathscr{I}\mathscr{S}_n(G)\} = \max\{\mathrm{cr}_n(G\backslash X) - n|X|\colon X \subseteq V(G)\}.$$

For $n = 1$ this is a well known result [14].

It is easy to see that the problem of finding an induced n-star packing which covers at least a given number of vertices in a graph belongs to the class NP. A natural question is whether this problem also belongs to CoNP. In other words we would like to know whether there exists (and we can find) a 'polynomial time' certificate for a graph not to have an induced n-star packing that covers a given number of vertices. The above theorem would provide such certificate if IS_n-critical graphs could be recognized in polynomial time. Therefore it is useful to have a good characterization of IS_n-critical graphs.

A graph is called IS-*critical* if it is IS_n-critical for every integer $n \geqslant 1$.

A graph F is called an *odd clique tree* if F is connected and every block of F is a complete graph with an odd number of vertices. We prove that

Theorem 2.2. *The following conditions are equivalent*:

(c1) *F is an* IS_n*-critical graph*, $n \geqslant 2$,

(c2) *F is a* IS*-critical graph, and*

(c3) *F is an odd clique tree.*

Let $\mathrm{oct}(G)$ denote the number of components of G which are odd clique trees. From Theorem 2.2 we have: $\mathrm{cr}_n(G) = \mathrm{oct}(G)$ for every integer $n \geqslant 2$, and so $\mathrm{cr}_n(G)$ does not depend on $n \geqslant 2$. Therefore from Theorems 2.1 and 2.2 we obtain a Duality Theorem that provides a good characterization of the problem.

Theorem 2.3. *Let $n \geqslant 2$. Then*

$$\min\{|W[A]|: A \in \mathscr{P}\mathscr{I}\mathscr{S}_n(G)\} = \max\{\mathrm{oct}(G\backslash X) - n|X|: X \subseteq V(G)\}.$$

This theorem is analogous to the Berge matching theorem [14].

From Theorems 2.2 and 2.3 it follows that the problem of finding an induced n-star packing that covers at least a given number of vertices in a graph belongs to CoNP.

The above Duality Theorem provides in particular a criterion for a graph to have a perfect induced n-packing which is analogous to the Tutte matching theorem [14].

Theorem 2.4. *A graph G has a perfect induced n-packing if and only if* $\mathrm{oct}(G\backslash X) \leqslant n|X|$ *for every* $X \subseteq V(G)$.

From Theorem 2.4 we have

Theorem 2.5. *A connected graph G does not have a perfect induced star packing if and only if G is an odd clique tree.*

The next theorem is analogous to the Gallai–Edmonds Structure theorem on matchings in a graph [14].

We use the following notation: $C_n(G)$ is the set of all vertices of G which are not covered by at least one V-maximum induced n-packing of G, $H_n(G)$ is the set of all vertices in $V(G)\backslash C_n(G)$ adjacent to at least one vertex in $C_n(G)$, and $D_n(G) = V(G)\backslash(H_n(G) \cup C_n(G))$. Let $\dot{D}_n(G)$ and $\dot{C}_n(G)$ denote subgraphs of G induced by the vertex sets $D_n(G)$ and $C_n(G)$, respectively.

A matching of G is called *nearly perfect* if it covers all but exactly one vertex of G.

Theorem 2.6. *Let G be a graph, and n be an integer at least* 2. *Then*

(a1) *the components of the subgraph $\dot{C}_n(G)$ are* IS_n*-critical* (*are odd clique trees*),

(a2) *the subgraph $\dot{D}_n(G)$ has a perfect induced n-packing,*

(a3) *if A is a maximum induced n-packing of G, then it contains*

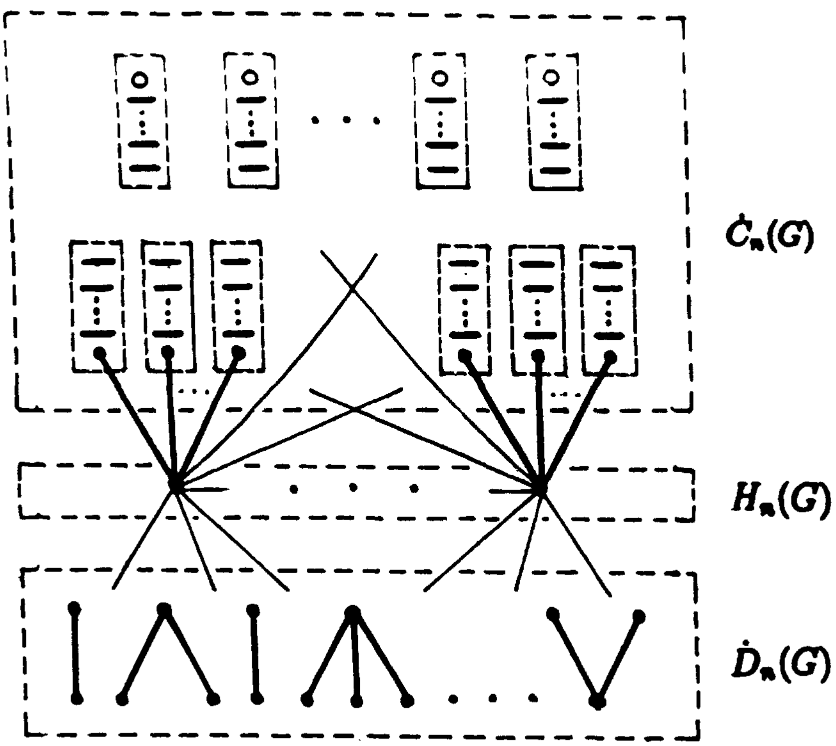

Fig. 1. The structure of a graph with respect to its induced n-packings.

(b1) *a near perfect matching of each component of the subgraph* $\dot{C}_n(G)$,

(b2) *a perfect induced n-packing of the subgraph* $\dot{D}_n(G)$, *and*

(b3) *a set of* $|H_n(G)|$ *disjoint n-stars such that their heads are in* $H_n(G)$, *their tails are in* $\dot{C}_n(G)$, *and each component of* $\dot{C}_n(G)$ *contains at most one tail of all these stars,*

(a4) $W[A] = \min\{|W[T]|: T \in \mathscr{P}\mathscr{I}\mathscr{S}_n(G)\} = \mathrm{oct}(G \backslash H_n(G)) - n|H_n(G)|$.

This theorem is illustrated in Fig. 1 where all rectangles in $\dot{C}_n(G)$ are all components (odd clique trees) of $\dot{C}_n(G)$, the thin lines are edges in $E(G)\backslash A$, and the thick lines are edges in A.

Let Z be a vertex subset of G such that $\min\{|W[A]|: A \in \mathscr{P}\mathscr{I}\mathscr{S}_n(G)\} = \mathrm{oct}(G\backslash Z) - n|Z|$. We call such subset Z an $\mathscr{I}\mathscr{S}_n$*-obstacle in* G because Z provides a tight upper bound on the number of vertices in G that can be covered by an induced n-packing. We prove the following theorem.

Theorem 2.7. *Let* X *and* Y *be* $\mathscr{I}\mathscr{S}$*-obstacles in* G. *Then* $X \cup Y$ *and* $X \cap Y$ *are also* $\mathscr{I}\mathscr{S}$*-obstacles in* G.

This theorem is similar to that for the matchings in a graph.

Let $\mathscr{X}_n(G)$ denote the set of all $\mathscr{IS}_n$-obstacles in G. By using Theorems 2.6 and 2.7 we prove

Theorem 2.8. *Let $\mathscr{L}_n(G) = (\mathscr{X}_n(G), \subseteq)$ denote the set $\mathscr{X}_n(G)$ partially ordered by the inclusion operation $\subseteq$. Then*

(a1) *$\mathscr{L}_n(G)$ is a sublattice of the lattice of all subsets of $V(G)$ under inclusion,*

(a2) *$H_n(G)$ is the minimum element of $\mathscr{L}_n(G)$, and*

(a3) *if $X \in \mathscr{L}_n(G)$ then $X \setminus H_n(G) \in \mathscr{L}_n(\dot{D}_n(G))$.*

An *augmenting A-trail* is a trail P of G such that the edges of A and $E(G)\setminus A$ alternate in P, $A \triangle E(P)$ is an induced n-packing, and $|V[A]| < V[A \triangle E(P)]$. We show that augmenting A-trails are sufficient tools to find a maximum induced n-packing.

Theorem 2.9. *An induced n-packing A of G covers the maximum number of vertices of G if and only if there is no augmenting A-trail in G.*

This theorem is similar to that for the matchings in a graph [14].

Let $\mathscr{IS}_n(G)$ denote the set of vertex subsets X of G such that X can be covered by an induced n-packing of G. It is easy to see that $\mathscr{IS}_n(G)$ has the hereditary property. Analysis of the properties of induced n-packings in a graph without augmenting A-trails enables us to give a description of a cycle of the hereditary family $\mathscr{IS}_n(G)$ of vertex sets of G (see **6.4** below). By using this description, we prove the following theorem.

Put $\mathscr{MIS}_n(G) = (V(G), \mathscr{IS}_n(G))$. An element of a matroid M is called *cyclic* if it belongs to at least one circuit of M, and *acyclic* or a *coloop* otherwise.

Theorem 2.10. *Let n be an integer, $n \geqslant 1$. Then*

(m1) *$\mathscr{MIS}_n(G)$ is a matroid with the independence set $\mathscr{IS}_n(G)$,*

(m2) *$C_n(G)$ is the set of cyclic elements of the matroid $\mathscr{MIS}_n(G)$, and*

(m3) *$H_n(G) \cup D_n(G)$ is the set of coloops of $\mathscr{MIS}_n(G)$.*

Note that $\mathscr{MIS}_1(G)$ is the well-known matching matroid of G [14].

Theorem 2.10 can also be proved by using Structure Theorem 2.6.

An edge subset A of G is called an *k-induced n-packings* if every component of $G[A]$ having at least $k+1$ edges is an induced subgraph of G.

Let $\mathscr{PIS}_n^k(G)$ denote the set of all k-induced n-packings of G, and let $\mathscr{IS}_n^k(G)$ denote the sets of vertex subsets X of G such that X is covered by a k-induced n-packing $k \geqslant 1$. In particular, $\mathscr{PIS}_n^1(G) = \mathscr{PIS}_n(G)$ is the set of all induced n-star packings of G, and $\mathscr{PIS}_n^k(G) = \mathscr{PS}_n(G)$ is the set of all n-star packings of G for $k \geqslant n$.

Theorem 2.11. *$\mathscr{IS}_n(G) = \mathscr{IS}_n^k(G)$ for every integer k and n such that $2 \leqslant k < n$.*

This theorem follows directly from the following simple claim.

Claim. *Let F be a graph having a vertex adjacent to every other vertex in F. Then F has a matching M such that if F is an odd clique then $F \backslash F[M]$ is a triangle, otherwise $F \backslash F[M]$ is an induced star in F.*

The approach developed below for obtaining the above results also gives natural proofs of the following results. Let $\mathscr{P}\mathscr{S}_n(G)$ denote the set of n-packings of G, and $\mathrm{isv}(G)$ denote the number of isolated vertices of G.

Theorem 2.12. *Let n be an integer, $n \geqslant 2$. Then*

$$\min\{|W[A]|\colon A \in \mathscr{P}\mathscr{S}_n(G)\} = \max\{\mathrm{isv}(G \backslash X) - n|X|\colon X \subseteq V(G)\}.$$

From the above theorem we have in particular

Theorem 2.13 (Hell and Kirkpatrick [5] and Las Vergnas [10]). *A graph G has a perfect n-packing if and only if*

$$\mathrm{isv}(G \backslash X) \leqslant n|X| \quad \textit{for every } X \subseteq V(G).$$

Let $\mathscr{S}_n(G)$ be the set of vertex subsets X of G such that X can be covered by an n-packing of G. Put $\mathscr{M}\mathscr{S}_n(G) = (V(G), \mathscr{S}_n(G))$.

Theorem 2.14 (Las Vergnas [10]). *$\mathscr{M}\mathscr{S}_n(G)$ is a matroid with the independence set $\mathscr{S}_n(G)$ for every integer $n \geqslant 1$.*

This theorem also follows immediately from the simple fact that $\mathscr{M}\mathscr{S}_n(G)$ is the union of n matching matroids $\mathscr{M}\mathscr{S}_1(G)$.

The same approach can be used to prove the following theorem on n-star packings analogous to the Gallai–Edmonds Structure theorem on matchings in a graph [14] and to Theorem 2.6 on induced n-star packings in a graph.

We use the following notation: $C'_n(G)$ is the set of all vertices of G which are not covered by at least one maximum n-packing of G, $H'_n(G)$ is the set of all vertices in $V(G) \backslash C'_n(G)$ adjacent to at least one vertex in $C'_n(G)$, $D'_n(G) = V(G) \backslash (H'_n(G) \cup C'_n(G))$. Let $\dot{D}'_n(G)$ and $\dot{C}'_n(G)$ denote subgraphs of G induced by the vertex sets $D'_n(G)$ and $C'_n(G)$, respectively.

Theorem 2.15. *Let G be a graph, and n be an integer at least 2. Then*

(a1) *the components of the subgraph $\dot{C}'_n(G)$ are S_n-critical (are isolated vertices),*

(a2) *the subgraph $\dot{D}'_n(G)$ has a perfect n-packing,*

(a3) *if A is a maximum n-packing of G, then it contains*

(b1) *a perfect n-packing of the subgraph $\dot{D}'_n(G)$, and*

(b2) *a set of $|H'_n(G)|$ disjoint n-stars such that their heads are in $H'_n(G)$, their tails are in $\dot{C}'_n(G)$,*

(a4) $W[A] = \min\{|V(G) \backslash V[Y]|\colon Y \in \mathscr{P}\mathscr{S}_n(G)\} = \mathrm{oct}(G \backslash H'_n(G)) - n|H'_n(G)|$.

Theorem 2.16. *Let G be a graph and $n \geqslant 1$ be an integer. Then $C_n'(G)$ and $H_n'(G) \cup D_n'(G)$ are the set of cyclic elements and the set of coloops of the matroid $\mathcal{MS}_n(G)$, respectively.*

By using arguments similar to that in our proofs, we can find polynomial-time algorithms for solving the corresponding problems (see Section 8). So we have

Theorem 2.17. *The following problems can be solved in polynomial time:*

(P1) *find an induced n-packing in a graph, covering the maximum number of vertices,*

(P2) *recognize* IS_n*-critical graphs,*

(P3) *recognize a graph having a perfect induced n-packing, and*

(P4) *recognize a vertex subset of G that can be covered by an induced n-star packing.*

It is known that some combinatorial optimization problems for matroids can be solved in a polynomial number of calls to the independence oracle of the matroid. By Theorem 2.17(P4), the independence oracle for the matroid $\mathcal{MIS}_n(G)$ can be realized by a polynomial-time algorithm. Therefore the corresponding combinatorial optimization problems for $\mathcal{MIS}_n(G)$ can be solved in polynomial time. In particular we have

Theorem 2.18. *The following problems can be solved in polynomial time:*

(P5) *find the maximum number of disjoint maximum induced n-packings in a graph,*

(P6) *find the minimum number of induced n-packings that covers all the vertices of a graph,*

(P7) *given a matroid M on V(G) find an induced n-packing of G that covers an independent set of M of maximum size, and*

(P8) *given non-negative weights of the vertices of G find an induced n-packing of G that covers a vertex set of G of maximum total weight.*

3. Alternating and augmenting trails

The concepts of *alternating* and *augmenting paths* play a very important role in the matching theory [14].

In 1971 we have introduced analogous concepts for a matroid which is the union of several matroids. We have considered special types of these paths (so called passive and active paths). We used the idea of a passive path reachability of a covered element from a non-covered element to give a simple algorithmic proof of the main theorem in matroid optimization on the rank of the union of matroids and to give a polynomial-time algorithms for solving packing, covering and intersection problems in matroids (described in [7–9]). In [8,9] this approach has been used to find a fastest known algorithm for solving these matroid optimization problems.

In this section we are going to introduce similar concepts of *alternating* and *augmenting paths* and in particular, *passive* and *active* paths and to use similar approach for analysing the induced n-star packing problem. By using passive alternating paths and the corresponding reachability, we will investigate the properties of a vertex maximal induced n-packing. The proofs of the main results described in the previous section are natural byproducts of this analysis.

A *trail* $P = x_1Px_k$ in a graph G is a sequence $x_1e_1x_2\dots x_{k-1}e_{k-1}x_k$ where each x_j is a vertex and each e_i is an edge of G, each $e_i = (x_i, x_{i+1})$, and $e_i \neq e_j$ for $i \neq j$. If in addition $x_i \neq x_j$ for $i \neq j$ then such trail is called a *path* in G. Since G has no loops or parallel edges, we can also describe a trail by the sequence of its vertices $P = x_1Px_k = x_1, x_2, \dots, x_k$. Note that $x_1Px_k = x_1, x_2, \dots x_{k-1}, x_k$ and $x_kPx_1 = x_k, x_{k-1}, \dots x_2, x_1$ are different paths. A *subtrail* x_iPx_j, $i \leqslant j$, of the trail x_1Px_k is a subsequence $x_ie_jx_{i+1}\dots e_{j-1}x_j$ of the sequence $x_1e_1x_2\dots x_{k-1}e_{k-1}x_k$. A *subpath* of a path is defined similarly.

Let A be an induced n-packing of G.

A trail $P = x_1e_1x_2\dots x_{k-1}e_{k-1}x_k$, $k \geqslant 2$, is called an *A-alternating trail* or simply an *A-trail* of G if $|\{e_i, e_{i+1}\} \cap A| = 1$ for every $i \in \{1, \dots, k-2\}$.

We recall that if X and Y are sets then $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

We are interested in A-alternating trails P such that if A is an induced n-packing then $A \triangle E(P)$ is also an induced n-packing in G.

An A-trail P is said to be *augmenting* if $A' = A \triangle E(P)$ is an induced n-star packing and $|V[A]| < |V[A']|$ (clearly in this case, $V[A] \subset V[A']$).

Let $\mathscr{S}(A)$ denote the set of stars of A, i.e. the set of components of $G[A]$. A star S of A is *big* if $|E(S)| = n$, *small* if $|E(S)| = 1$, and *intermediate* if $1 < |E(S)| < n$. A *k-star* is a star with k edges. Let A_k denote the set of edges in A belonging to the k-stars of A, i.e. $A_k = \bigcup \{E(S)\colon S \in \mathscr{S}_k(A)\}$. Small and big stars and, respectively, A_1 and A_n will play an essential role in our consideration. If S is a non-small star then the common vertex of all edges of S is called the *head* of S and a vertex of S distinct from the head of S (a vertex of degree 1 in S) is called a *tail* of S. Let $h(S)$ and $T(S)$ denote the head and the set of tails of S, respectively.

An A-trail $x_1Px_k = x_1e_1x_2\dots x_{k-1}e_{k-1}x_k$, $k \geqslant 2$, is called a *passive A-path* if the following conditions hold:

(p0) P is an A-path,

(p1) $x_1 \in W[A] = V(G) \setminus V[A]$,

(p2) the last edge e_{k-1} of P belongs to A: $e_{k-1} \in A$, so that k is an odd integer, and $e_i \in A$ for every even integer $i < k$,

(p3) every edge in $A \cap E(P)$ belongs to either a big star or a small star of A,

(p4) if $e_i = (x_i, x_{i+1}) \in A_n$ (i.e. e_i is an edge of a big star, say S^i, of A), then x_i is the head, x_{i+1} is a tail of S^i, and x_{i-1} is adjacent to no tail of S^i in G: $x_i = h(S^i)$, $x_{i+1} \in T(S^i)$, $(x_{i-1}, t) \notin E(G)$ for $t \in T(S^i)$, and

(p5) if $e_i = (x_i, x_{i+1}) \in A_1$ (i.e. e_i is the edge of a small star S of A), then $e'_{i-1} = (x_{i-1}, x_{i+1})$ (as well as $e_{i-1} = (x_{i-1}, x_i)$) is an edge of G.

An example of a passive A-path, $n = 3$, and its alternation are shown in Fig. 2.

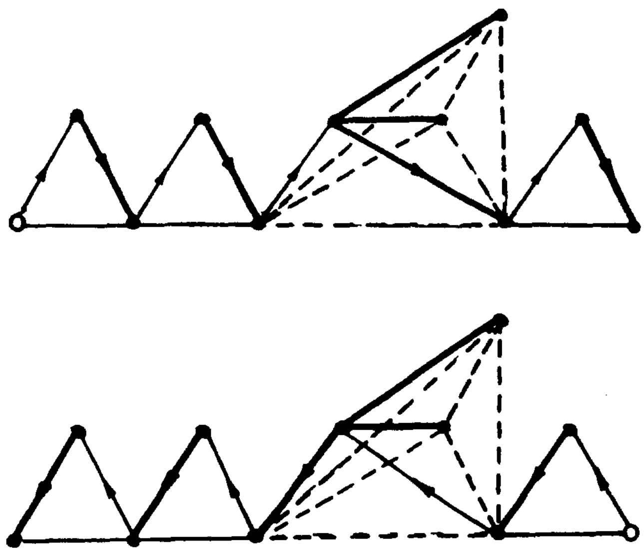

Fig. 2. An example of a passive A-path, $n = 3$, and its alternation.

In this and every figure below a thin line is an edge in $E(G)\backslash A$, a thick line is an edge in A, and a broken line is a non–edge of G.

A path consisting of one vertex in $W[A]$ is a *trivial passive A-path.*

It is easy to see that

3.1. *Let A be an induced n-packing of G, and $x_1 P x_k = x_1, x_2, \ldots, x_{k-1}, x_k$, $k \geqslant 3$, be a passive A-path. Then*

(s1) *if i is odd (i.e. $e_{i-1} \in A$) then the subpath $x_1 P x_i$ of $x_1 P x_k$ is a passive A-path,*

(s2) *for every vertex z of a passive path xPy there exists a subpath $x_1 P x_{i+1}$ of $x_1 P x_k$ such that $x_1 P x_{i+1}$ is a passive A-path and z is an end-vertex of the last edge $e_i = (x_i, x_{i+1})$ of $x_1 P x_{i+1}$: $z \in \{x_i, x_{i+1}\}$,*

(s3) *if x_i is the head of a big star of A, then i is even,*

(s4) *if i is even and x_i is not the head of a big star (i.e. (x_i, x_{i+1}) form a small star of A and $(x_{i-1}, x_{i+1}) \in E(G)\backslash A$) then the path $x_1 P x_{i-1}, x_{i+1}, x_i$ is a passive A-path.*

Clearly passive A-paths have the following important 'exchange' property.

3.2. *Let A be an induced n-packing of G, and let xPy be a passive A-path. Then*

(s1) *$A' = A \triangle E(P)$ is an induced n-packing of G, and $V[A'] = V[A] \backslash y \cup x$, so that $|V[A']| = |V[A]|$, and xPy is not an augmenting A-path, and*

?) *yPx is a passive A'-path.*

words, a passive A-path P allows us to move from a given induced nother induced n-packing A' whose vertex set $V[A']$ is obtained from

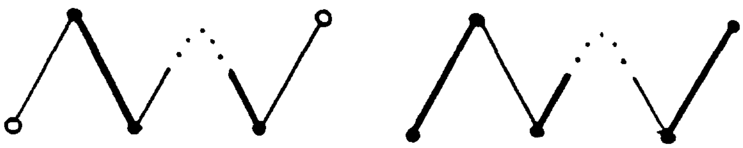

Fig. 3. A 1-active A-path and its alternation.

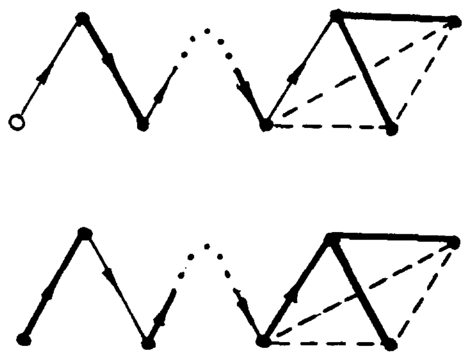

Fig. 4. A 2-active A-path and its alternation.

the vertex set $V[A]$ by substituting the last vertex of P by the first vertex of P (see Fig. 2).

Note that the requirment $(x_{i-1}, t) \notin E(G)$ for $t \in T(S^i)$ in (p4) is essential. For if (x_{i-1}, t) is an edge for some $t \in T(S^i)$, then $A' = A \triangle E(P)$ is not an induced n-packing, namely the component of $G[A']$ containing $S^i \setminus e_i$ is not an induced star (and also in this case the subpath $x_1 P x_{i-1}$ of P can be extended to the augmenting path $x_1 P x_{i-1}, t, h(S^i)$).

Now we will consider some special augmenting A-trails (so called *active A-trails*).

An A-trail $P = x_1 e_1 x_2 \ldots x_{k-1} e_{k-1} x_k$, $k \geqslant 2$, is called a 1-*active A-path* if

(p1.1) P is a path,

(p1.2) the subpath $x_1 P x_{k-1}$ of P is a passive A-path (and so $x_1 \in W[A]$), and

(p1.3) $x_k \in W[A]$ (see Fig. 3).

An A-trail $P = x_1 e_1 x_2 \ldots x_{k-1} e_{k-1} x_k$, $k \geqslant 2$, is called a 2-*active A-path* if

(p2.1) P is a path,

(p2.2) the subpath $x_1 P x_{k-1}$ of P is a passive A-path (and so $x_1 \in W[A]$), and

(p2.3) x_k is the head of an intermediate star, say S, and x_{k-1} is adjacent to no tail of S in G (see Fig. 4).

An A-trail $P = x_1 e_1 x_2 \ldots x_{k-1} e_{k-1} x_k$, $k \geqslant 2$, is called a 3-*active A-path* if

(p3.1) P is a path,

(p3.2) the subpath $x_1 P x_{k-1}$ of P is a passive A-path (and so $x_1 \in W[A]$), and

(p3.3) x_k is a vertex of a small star, say (x_k, y), and (x_{k-1}, y) is not an edge of G (see Fig. 5).

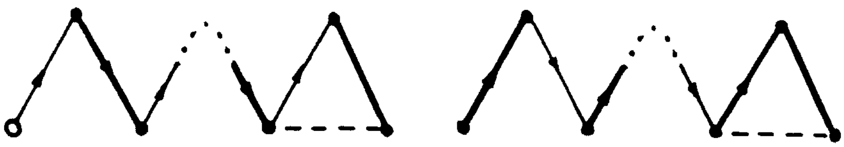

Fig. 5. A 3-active A-path and its alternation.

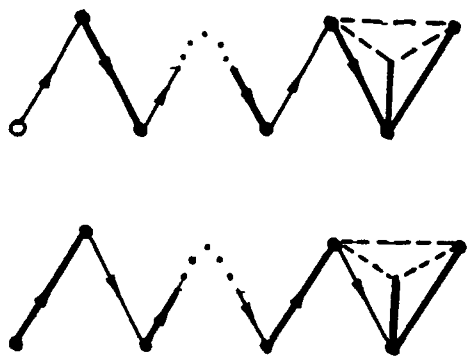

Fig. 6. A 4-active A-path and its alternation.

An A-trail $P = x_1e_1x_2 \ldots x_{k-1}e_{k-1}x_k$, $k \geqslant 3$, is called a *4-active A-path* if

(p4.1) P is a path,

(p4.2) the subpath x_1Px_{k-2} of P is a passive A-path (and so $x_1 \in W[A]$), and

(p4.3) x_{k-1} and x_k are a tail and the head of some non-small star Z of A, respectively, i.e. $x_{k-1} \in T(Z)$ and $x_k = h(Z)$ (see Fig. 6).

An *active A-path* is an s-active A-path for some $s = 1,2,3,4$.

In other words, the only way to create an active A-path in G is to add to a passive A-path Q a new edge e of G connecting the last vertex l of P either with a vertex in $W[A]$ distinct from the first vertex of P, or with the head of an intermediate star of A, or with a vertex of a small star Z of A if e is the only edge between l and Z in G, or with a tail of a non-small star of A. In particular, an edge with both end vertices in $W[A]$ is a active A-path; we call such path a *trivial active A-path.*

A trail $xPx'ey$ is called a *quasi-path* if P is a path and $y \in V(P)$.

An A-trail $x_1Px_k = x_1e_1x_2 \ldots x_{k-1}e_{k-1}x_k$, $k \geqslant 6$, is a *1-active A-quasi-path* if

(q1.1) the subtrail x_1Px_{k-1} is a passive A-path (and so $x_1 \in W[A]$),

(q1.2) $x_k = x_{k-4}$,

(q1.3) $(x_{k-5}, x_{k-1}) \notin E(G)$,

(q1.4) $e_{k-4} = (x_{k-4}, x_{k-3})$ and $e_{k-2} = (x_{k-2}, x_{k-1})$ form small stars of A, and

(q1.5) the 4-vertex set $\{x_k = x_{k-4}, x_{k-3}, x_{k-2}, x_{k-1}\}$ induces a complete subgraph, say K, in G, and so $E(K) \setminus \{e_{k-4}, e_{k-2}\} \subseteq E(G) \setminus A$ (see Fig. 7).

An A-trail $x_1Px_k = x_1e_1x_2 \ldots x_{k-1}e_{k-1}x_k$, $k \geqslant 6$, is a *2-active A-quasi-path* if

(q2.1) the subtrail x_1Px_{k-2} is a passive A-path (and so $x_1 \in W[A]$),

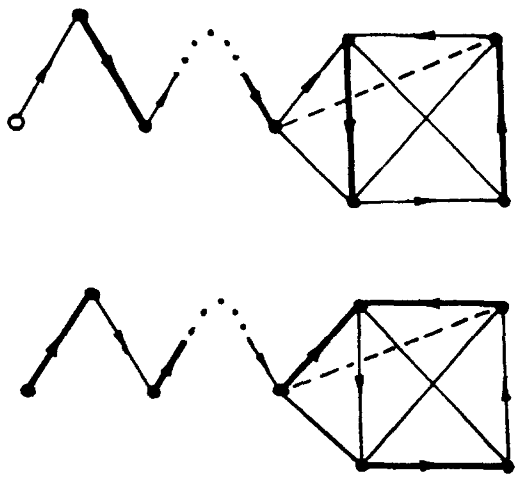

Fig. 7. A 1-active A-quasi-path and its alternation.

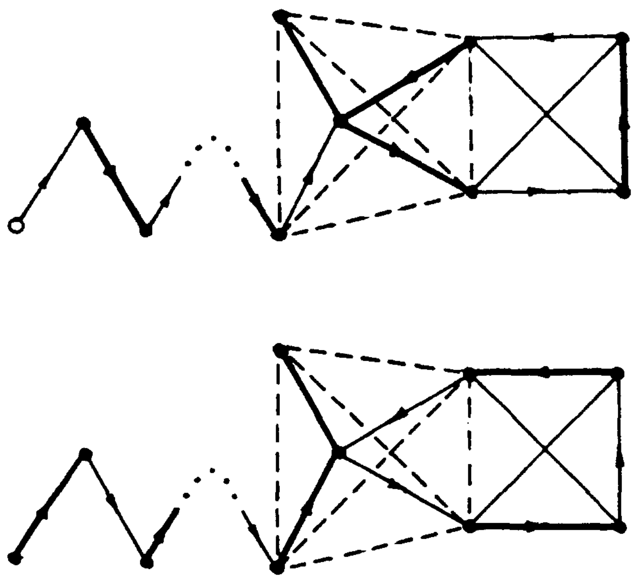

Fig. 8. A 2-active A-quasi-path and its alternation.

(q2.2) $e_{k-5} = (x_{k-5}, x_{k-4})$ is an edge of a big star, say S, of A, so that $x_{k-5} = h(S)$ and $x_{k-4} \in T(S)$,

(q2.3) $x_k = x_{k-5}$, so that $x_k = h(S)$, $x_{k-1} \in T(S) \backslash x_{k-4}$, and $(x_{k-4}, x_{k-1}) \notin E(G)$,

(q2.4) the edge $e_{k-3} = (x_{k-3}, x_{k-2})$ forms a small star of A,

(q2.5) $(x_{k-4}, x_{k-2}), (x_{k-3}, x_{k-1}) \in E(G) \backslash A$ (see Fig. 8).

An *active A-quasi-path* is an s-active A-quasi-path for some $s = 1, 2$. Clearly an active A-quasi-path is a quasi-path.

An *active A-trail* is an active A-path or an active A-quasi-path.

It is easy to see that active A-trails have the following important properties.

3.3. *Let A be an induced n-packing of G. Then*

(s1) *if xPy is a passive A-path then xQz where $Q = P\setminus y$ is an active A^y-path in $G\setminus y$ where $A^y = A \cap E(G\setminus y)$, and*

(s2) *if xPy is an active A-path or an active A-quasi-path, then $A' = A \triangle E(P)$ is an induced n-packing of G, and $V[A] \subset V[A']$, i.e. an active A-path is an augmenting A-path and an active A-quasi-path is an augmenting A-quasi-path.*

4. Induced *n*-star packings without augmenting trails

We say that an induced n-packing A of G is *vertex maximal* or simply *V-maximal* if there is no induced n-packing A' of G such that $V[A] \subset V[A']$.

From **3.3** we have

4.1. *Let A be a V-maximal induced n-packing of G. Then*

(s1) *there is no augmenting A-trail, in particular,*

(s2) *there is no active A-path and no active A-quasi-path in G, and in particular,*

(s3) *there is no edge having both its end-vertices in $W[A]$.*

We will use the following notation: B^A is the set of vertices of G that can be reached from $W[A]$ by a passive A-path, $\dot{B}^A$ is the subgraph of G induced by B^A (i.e. $\dot{B}^A = G[B^A]$), S^A is the set of big stars of A in $\dot{B}^A$, $T^A = \bigcup\{T(S)\colon S \in \mathcal{S}^A\} \cup W[A]$), H^A is the set of heads of big stars in $\dot{B}^A$ (i.e. $H^A = \{h(S)\colon S \in \mathcal{S}^A\}$), $C^A = B^A \setminus H^A$, $D^A = V(G)\setminus B^A$, $\dot{D}^A = G[D^A]$, $C^A = B^A\setminus H^A$ and $\dot{C}^A = G[C^A] = \dot{B}^A \setminus H^A$.

4.2. *Let A be an induced n-packing of G. Let S be a star of A. Then*

(a1) *if S is an intermediate star, then $V(S) \cap B^A = \emptyset$ (and so $S \cap \dot{B}^A = \emptyset$), and*

(a2) *if $V(S) \cap B^A \neq \emptyset$ then $S \subseteq \dot{B}^A$.*

Proof. (p1) Let S be an intermediate star of A. Suppose on the contrary that $Z = V(S) \cap B^A \neq \emptyset$. Then for every vertex x in Z there exists a passive A-path containing x. Let P be a shortest passive A-path containing a vertex in Z. Then the last edge of P is an edge of S. By definition of a passive A-path, an edge of the passive A-path P cannot belong to the intermediate star S, a contradiction.

(p2) Let S be a star of A, and let $Z = V(S) \cap B^A \neq \emptyset$. Let $z \in Z$. By **(p1)**, S is not an intermediate star, i.e. S is either a small star or a big star of A. By the definition of $\dot{B}^A$, there exists a passive A-path $x_1Px_k = x_1e_1 \dots e_{k-1}x_k$ containing z. Then x_1Px_k contains an edge, say $e = (x_i, x_{i+1})$, of S incident to z: $e \in E(S)$. Therefore e is a common edge of S and $\dot{B}^A$. Now if S is a small star, then $S \subseteq \dot{B}^A$, and we are done. Therefore let S be a big star. Then $h(S) = x_i$. Since x_1Px_k is a passive A-path, it follows from the definition of a passive A-path that for every tail t of S the path x_1Px_i, t is a passive A-path. Therefore $S \subseteq \dot{B}^A$. □

From **4.2** we have

4.3. *Let A be an induced n-packing of G. Then $A \cap \dot{D}^A$ covers all the vertices of $\dot{D}^A$ and every star of $A \cap \dot{D}^A$ is a star of A.*

From 3.1 (s1) and (s4) we have

4.4. *Let A be an induced n-packing of G. Then C^A is the set of the last vertices of all passive A-paths of G.*

4.5. *Let A be an induced n-packing of G. Suppose that*
(h1) *$P := x_1Px_k = x_1, x_2, \dots, x_k$ is an A-path,*
(h2) *$x_1 \in W[A]$,*
(h3) *P has an even number of edges so that $(x_{k-1}, x_k) \in A$,*
(h4) *P is not a passive A-path.*
Then G has an active A-path starting at x_1.

Proof. Since x_1 is a (trivial) passive A-path, and $x_1 \in P$, the path x_1Px_k has passive A-subpaths. Let x_1Px_m be a (the) maximal passive A-subpath of x_1Px_k (so that $(x_{m-1}, x_m) \in A$). Since x_1Px_k is not a passive A-path, we have: $1 < m \leqslant k-2$. Let us consider the subpath $x_1Px_{m+2} = x_1Px_m, x_{m+1}, x_{m+2}$. Clearly $e_{m+1} \in A$.

Suppose that the edge (x_{m+1}, x_{m+2}) form a small star of A. If $(x_m, x_{m+2}) \in E(G)$ then x_1Px_{m+2} is a passive A-path of P, and $x_1Px_m \subset x_1Px_{m+2}$. Therefore the passive subpath x_1Px_m of P is not maximal, a contradiction. If $(x_m, x_{m+2}) \notin E(G)$ then x_1Px_{m+2} is a 3-active A-path.

Now suppose that (x_{m+1}, x_{m+2}) is an edge of a non-small star S of A. If x_{m+1} is a tail of S then x_1Px_{m+2} is a 4-active A-path. Therefore let x_{m+1} be the head of S. If $(x_m, t) \in E(G)$ for some $t \in T(S)$ then x_1Px_m, t, x_{m+1} is a 4-active A-path. Therefore let $(x_m, t) \notin E(G)$ for every $t \in T(S)$. If S is an intermediate star, then x_1Px_{m+1} is a 2-active A-path. If S is a big star then x_1Px_{m+2} is a passive A-subpath of P, containing properly x_1Px_m. Therefore the passive A-subpath x_1Px_m of P is not maximal, a contradiction. □

4.6. *Let A be an induced n-packing of G. Suppose that*
(h1) *$x_1Px_k = x_1, x_2, \dots, x_k$ is an A-path, and*
(h2) *$x_1, x_k \in W[A]$.*
Then G has an active A-path starting at x_i, $i = 1, k$.

Proof. Let (as in the previous proof) x_1Px_m be a (the) maximal passive A-subpath of x_1Px_k (so that $(x_{m-1}, x_m) \in A$). If $m < k-1$ then by 4.5, G has an active A-path starting at x_1. Therefore let $m = k - 1$, and so x_1Px_{k-1} is a passive A-path. Since $x_1, x_k \in W[A]$, clearly x_1Px_k is an 1-active A-path. Since x_kPx_1 is also an A-path, the same is true for x_k. $\square$

Let $\bar{B}^A$ be obtained from $\dot{B}^A$ by deleting the edges connecting vertices in H^A, i.e. $\bar{B}^A = \dot{B}^A \setminus E(G[H^A])$. From **4.6** we have

4.7. *Let A be an induced n-packing of G. Suppose that G has no active A-path. Then $\bar{B}^A$ has exactly $|W[A]|$ components each containing one vertex from $W[A]$.*

From the proofs of **4.5** and **4.6** we have

4.8. *Let the assumption of either* **4.5** *or* **4.6** *holds. Then G has an active A-path R such that*

(a1) *R contains the maximal passive A-path P' of P, and moreover*

(a2) *R is obtained from P' by one of the four operations described in the definition of an active A-path, so that either $R \subseteq P$ or $R \subseteq G[P \cup t]$ where t is a tail of a big star whose head is the vertex of P' adjacent to the last vertex of P'.*

An A_1-*subpath* xPy is a subpath of a passive A-path such that

(b1) every odd edge is in $E(G)\setminus A$, in particular, the first edge (x, x') is in $E(G)\setminus A$,

(b2) every even edge is in A and forms a small star of A, and

(b3) the last edge is in A (and so xPy has an even number of edges).

Since P is a subpath of a passive A-path, every small star $G[x_i, x_{i+1}]$ in P is connected by two edges with the previous vertex of P: $(x_{i-1}, x_i), (x_{i-1}, x_{i+1}) \in E(G)\setminus A$.

4.9. *Let A be an induced n-packing of G. Suppose that*

(h1) *$P := x_1Px_k = x_1, x_2, \dots, x_k$ is a passive A-path (so that $x_1 \in W[A]$),*

(h2) *x_{k-1} is the head of a big star S: $x_{k-1} = h(S)$ and $x_k \in T(S)$,*

(h3) *t_1Qt_2 is an A_1-subpath and $t_1, t_2 \in T(S)$, and*

(h4) *$x_1Px_{k-1} \cap Q = \emptyset$.*

Then $G[P \cup Q]$ has an active A-trail R of G containing $P\setminus x_k$, and R is either a 3-active A-path, or s-active A-quasi-path, $s = 1, 2$.

Proof. Let $t_1Qt_2 = (t_1 = y_1), y_2 \dots y_{s-1},\ (y_s = t_2)$. Then $(y_1, y_2), (y_{s-1}, y_s) \in E(G)\setminus A$. Put $h := h(S) = x_{k-1}$. Since x_1Px_k is a passive A-path, also x_1Px_{k-1}, t is a passive A-path for every $t \in T(S)$. In particular, $x_1P_it_i = x_1Px_{k-1}, t_i$ is a passive A-path.

Let R_1 be a (the) maximal passive A-subpath of $x_1P_1t_1Qt_2$. Clearly $x_1P_1t_1 \subseteq R_1$. Let $R_1 = x_1P_1t_1Qy_m$.

If $R_1 \neq P_1 \cup Q\setminus t_2$, i.e. $m < s-1$ then clearly $x_1R_1y_m, y_{m+1}, y_{m+2}$ is an 3-active A-path in $G[P \cup Q]$ containing $P\setminus x_k$.

Therefore we can assume that $R_1 = P_1 \cup Q\setminus t_2$. By the same reason, we can assume that $R_2 = P_2 \cup Q\setminus t_1$ is the maximal passive A-subpath of the A-path $P_2 \cup Q$. Thus $Q\setminus t_i$ is a A_1-subpath for $i = 1, 2$.

Let us prove that if $(t_1, y_i) \notin E(G)$ for some $i \in \{4,\dots,s-1\}$, then $G[P \cup Q]$ has a 1-active A-quasi-path containing P. Let k be the minimum integer such that $(t_1, y_k) \notin E(G)$.

Suppose that k is even. Then $(y_{k-2}, y_{k-1}) \in A_1$ and $(y_k, y_{k+1}) \in A_1$, and also $(t_1, y_{k-2}), (t_1, y_{k-1}) \in E(G)\backslash A$. Clearly $(y_{k-1}, y_k) \in E(G)\backslash A$. Since $Q\backslash t_2$ is an A_1-subpath starting at t_1, we have $(y_{k-1}, y_{k+1}) \in E(G)\backslash A$. Since $Q\backslash t_1$ is an A_1-subpath starting at t_2, we have $(y_{k-2}, y_k) \in E(G)\backslash A$. If $(y_{k-2}, y_{k+1}) \notin E(G)$ then $P_1 y_1, y_{k-1}, y_{k-2}, y_k, y_{k+1}$ is a 3-active A-path in $G[P \cup Q]$ containing $P\backslash x_k$. Therefore we can assume that $(y_{k-2}, y_{k+1}) \in E(G)$. Thus $G[y_{k-2},\dots y_{k+1}]$ is a complete graph K_4. Let $L = t_1, y_{k-2}, y_{k-1}, y_{k+1}, y_k, y_{k-2}$. Then $P \cup L$ is a 1-active A-quasi-path in $G[P \cup Q]$ containing $P\backslash x_k$.

If k is odd then similar arguments show that G has a required active A-trail.

Thus we can assume that $(t_1, y_i) \in E(G)$ for every $i \in \{1,\dots,s-1\}$, and in particular, for $i = s-2, s-1$. Let $L = (t_1 = y_1), y_{s-2}, y_{s-1}, (y_s = t_2)L_2 x_k$. Then $P_1 \cup L$ is a 2-active A-path in $G[P \cup Q]$ containing $P\backslash x_k$. □

4.10. *Let A be an induced n-packing of G. Suppose that*

(h1) *$u_1 L_1 v_1$ and $u_2 L_2 v_2$ are two A_1-subpaths,*

(h2) *$u_1 \notin L_2$ and $u_2 \notin L_1$, and*

(h3) *either $L_1 \cap L_2 \neq \emptyset$ or $L_1 \cap L_2 = \emptyset$ and there exists an edge of G connecting a vertex of L_1 with a vertex of L_2.*

Then $G[L_1 \cup L_2]$ has an A-path connecting the vertices u_1 and u_2.

Proof. (p1) Suppose that $L_1 \cap L_2 = \emptyset$ and $e = (z_1, z_2) \in E(G)$ where $z_1 \in V(L_1)$ and $z_2 \in V(L_2)$. Since $u_i L_i v_i$ is a passive A_1-subpath, $G[L_i]$ has an A_1-subpath $u_i Q z_i$ with the end vertex z_i, $i = 1,2$. Then $u_1 Q_1 z_1 e z_2 Q_2 u_2$ is an A-path in $G[L_1 \cup L_2]$ connecting u_1 and u_2.

(p2) Suppose that $L_1 \cap L_2 \neq \emptyset$. Let x be the first vertex of L_1 that belongs to L_2 (if we walk along L_1 from u_1). Let $L_i = (y_i, x)$ be the edge of L_i which is incident to x and which does not belong to L_j, $i \neq j$, $\{i,j\} = \{1,2\}$. Clearly at most one of the edges l_1, l_2 belongs to A (and therefore to A_1). So let $l_1 \notin A_1$. Since $u_2 L_2 v_2$ is a A_1-subpath, $G[L_2]$ has a A_1-subpath $u_2 Q_2 x$. Since $u_1 L_1 x \cap G[L_2] = x$, clearly $u_1 L_1 x Q_2 u_2$ is an A-path in $G[L_1 \cup L_2]$ connecting u_1 and u_2. □

4.11. *Let A be an induced n-packing of G. Let xRy and $x'R'y'$ be two maximal A_1-subpaths in G. Suppose that*

(h1) *G has no active A-trail, and*

(h2) *$x \neq x'$.*

Then $R \cap R' = \emptyset$ and there is no edge of G connecting R with R'.

Proof (Uses **4.6**, **4.9** and **4.10**). Let xRy be a maximal A_1-subpath in G. Clearly $x \in T^A$. Let $\mathscr{R}$ denote the union of all maximal passive A_1-subpaths $x'R'y'$ with $x' \neq x$. Clearly it is sufficient to prove that $R \cap \mathscr{R} = \emptyset$ and there is no edge in G connecting R

with $\mathscr{R}$. Suppose the contrary. Let r be the first vertex of R that does not belong to $\mathscr{R}$ and is linked with a vertex, say r', in $\mathscr{R}$ by an edge, say $e = (r, r')$. Since $x \notin \mathscr{R}$, such vertex r exists. Let $x'R'y'$ be a maximal A_1-subpath in G such that $r' \in R'$ and $r' \neq r$. Clearly $x' \in T^A$. If $e = (r, r') \in A$ then $e \in R'$, and so $r \in R'$. But $r \notin R'$, a contradiction. Therefore $e \notin A$. Since $x'R'y'$ is a passive A-subpath, there exists a passive A-path $wP'x'R'y'$ containing R'. By the property of the vertex r, we have $xRr \cap (P' \cup R') = \emptyset$, and in particular, $xRr \cap R' = \emptyset$. By **4.10**, $G[R \cup R']$ has an A_1-subpath xQx'. Since $G[R \cup R'] \cap P' = x'$, clearly $Q \cap P' = x'$. Therefore $wP'x'Qx$ is an A-path starting at $w \in W[A]$ and terminating at x. We recall that $x \in T^A$.

Suppose that $x \in W[A]$. Then $x \neq w$. By **4.6**, G has an active A-path starting at x, a contradiction.

Now suppose that x is a tail of a big star S: $x \in T(S)$. Let $h = h(S)$, and $f = (h, x)$.

Suppose that $x' \notin T(S)$. Then $wTxfh = wDh$ is an A-path. Since the last vertex h of wDh is the head of a big star and the last edge of wDh is an edge of a big star, D is not a passive A-path. Since $w \in W[A]$, the A-path wDh satisfies the hypothesis of **4.5**. Therefore by **4.5**, G has an active A-path starting at w, a contradiction.

Now suppose that $x' \in T(S)$. We recall that $x \in T(S)$ and $x \neq x'$. Then by **4.9**, G has either an active A-path or an active A-quasi-path, a contradiction. □

An *A_1-cycle* Q of G is a cycle obtained from an A_1-subpath, say xLy, of G by adding the edge of G connecting the end vertices of the A_1-subpath L: $Q = L \cup e$ where $e = (x, y) \in E(G)$. It is clear that

4.12. *If an A_1-cycle has an even number of vertices, then $A_1 \cap Q$ is a perfect matching of Q. If an A_1-cycle has an odd number of vertices, then $A_1 \cap Q$ is a perfect matching of $Q \backslash z$ for some vertex z of Q.*

4.13. *Let A be an induced n-packing of G. Suppose that*

(h1) *$x_1Px_k = x_1, x_2, \ldots, x_k$ is a passive A-path (so that $x_1 \in W[A]$),*

(h2) *Q is an A_1-cycle, and*

(h3) *one of the following holds:*

(h3.1) *Q has an odd number of edges and $P \cap Q = x_k$, and*

(h3.2) *Q has an even number of edges, $P \cap Q = \emptyset$ and $(x_k, t) \in E(G)$ where $t \in V(Q)$.*

Then one of the following conditions holds:

(c1) *$G[P \cup Q]$ has an active A-trail containing P, and this A-trail is either a 3-active A-path or a 1-active A-quasi-path, and*

(c2) *$G[Q \cup x_k]$ is a complete graph.*

Proof (Uses **4.5** and **4.8**). Let $Q = (t = y_1)f_1y_2 \ldots f_{k-1}(y_k = t)$ where in case (h3.1); $t = x_k$ and so $Q \cup X_k = Q$. We prove the statement for case (h3.1). The proof for case (h3.2) is similar.

(p1) Let us consider an A-path $R_1 = x_1Px_kQy_{k-1} = P \cup Q \backslash f_{k-1}$. Clearly $x_1 \in W[A]$ and R_1 has an even number of vertices.

Suppose that R_1 is not a passive A-path. Then by **4.5**, G has an active A-path. Moreover, since P is a passive A-path and $Q \backslash f_{k-1}$ is an A_1-subpath, it follows from **4.5** and **4.8** that $G[R_1]$ has a 3-active A-path containing P.

Thus we can assume that $R_1 = P \cup Q \backslash f_{k-1}$ is a passive A-path. Therefore $(y_i, y_{i+2}) \in E(G) \backslash A$ for every odd $i \in \{1, \dots, k-3\}$. Similarly we can assume that $R_2 = P \cup Q \backslash f_1$ is a passive A-path. Therefore $(y_{i+2}, y_i) \in E(G)$ for every even $i \in \{2, \dots, k-2\}$.

Suppose that $(y_j, y_{j+3}) \notin E(G)$ for some even $j \in \{2, \dots, k-4\}$. We can assume that j is the minimum number for which it occurs. Then $PtQy_{j-1}, y_{j+1}, y_j, y_{j+2}, y_{j+3}$ is a 3-active A-path in $G[P \cup Q]$ containing P.

Now suppose that $(y_i, y_{i+3}) \in E(G)$ for every even $i \in \{2, \dots, k-4\}$.

Put $D_i = G[y_i, \dots, y_{i+3}]$ for every even $i \in \{2, \dots, k-4\}$. By the above arguments, D_i is a complete graph K_4.

(p2) Let us prove by induction on $s = |A_1 \cap Q|$ the following claim:

Claim. *If $G[P \cup Q]$ has no 1-active A-quasi-path containing P, then $G[Q]$ is a complete graph.*

If $s \leqslant 2$ then the statement follows from **(p1)**. Let $s \geqslant 3$. If $(t, y_4) \notin E(G)$ then put $L_4 = t, y_2, y_3, y_5, y_4, y_2$. If $(t, y_5) \notin E(G)$ then put $L_5 = t, y_3, y_2, y_4, y_5, y_3$. Then $P \cup L_i$, $i = 4$, is a 1-active A-quasi-path containing P, a contradiction. Therefore $(t, y_i) \in E(G) \backslash A$ for $i = 4, 5$. By similar arguments, $(t, y_i) \in E(G)$ for $i = k-4, k-3$. Put $Q^1 = Q \backslash \{y_2, y_3\} \cup (t, y_4)$, $Q^2 = Q \backslash \{y_{k-2}, y_{k-1}\} \cup (t, y_{k-4})$, and $Q^3 = Q \backslash \{y_4, y_5\} \cup (y_3, y_6)$. Then Q^i is a closed A_1-subpath, $Q^i \cap P = t$, and $|A_1 \cap Q^i| = s - 1$ for $i = 1, 2, 3$.

By the inductive hypothesis, $G[Q^i]$ is a complete graph for $i = 1, 2, 3$. Therefore $G[Q]$ is a complete graph. □

Remark. The above statement can also be proved by induction on $|A_1 \cap Q|$, by contracting a triangle containing x_k and a small star of Q.

From **4.11** and **4.13** we have

4.14. *Let A be an induced n-star packing in G. Suppose that*

(h1) *G has no active A-trail,*

(h2) *Q is an A_1-cycle, and*

(h3) *there exists $a \in T^A$ such that either $Q \cap T^A = \emptyset$ and $(a, q) \in E(G)$ or $Q \cap T^A = a$ for some $q \in V(Q)$.*

Then $K = G[Q \cup a]$ is a complete graph, and $A_1 \cap K$ is a perfect matching of $K \backslash a$.

Given $a \in T^A$ let Y^a denote the union of all A_1-subpaths starting at a. Let $N^a = G[Y^a]$. In other words, N^a is the subgraph of G induced by the set of vertices that can be reached from the vertex a by a A_1-subpath. We call N^a an A_1-*subgraph* of G. It easy to see that

4.15. *Let A be an induced n-packing of G, and let $a \in T^A$. Then*
(a1) *N^a is a connected subgraph of G, and*
(a2) *$A \cap N^a$ is a perfect matching of $N^a \backslash a$ (and so N^a has an odd number of vertices).*

From **4.11** we have

4.16. *Let A be an induced n-packing of G. Let $a, b \in T_A$ and $a \neq b$. Suppose that G has no active A-trail. Then $N^a \cap N^b = \emptyset$, and G has no edge connecting a vertex in N^a with a vertex in N^b.*

Let $\mathrm{Cmp}(F)$ denote the set of connected components of a graph F. By the definition of $\dot{C}^A$, for every vertex x of $\dot{C}^A$ there exists $a \in T^A$ such that $x \in N^a$. Therefore from **4.15** and **4.16** we have

4.17. *Let A be an induced n-packing of G. Suppose that G has no active A-trail. Then* $\mathrm{Cmp}(\dot{C}^A) = \{N^a\colon a \in T^A\}$.

4.18. *Let A be an induced n-packing of G. Suppose that G has no active A-trail. Then G has no edge connecting a vertex in C^A with a vertex in D^A.*

Proof (Uses **4.3**–**4.5**). Suppose on the contrary, that there exists an edge (c, d) of G such that $c \in C^A$ and $d \in D^A$. By **4.4**, c is the end vertex of a passive path, say wPc, and so $w \in W[A]$ and c is either a tail of a big star or an end vertex of a small star. Therefore $(c, d) \in E(G)\backslash A$. By **4.3**, every vertex of D^A belongs to a star of A which is a subgraph of $\dot{D}^A$. Let (d, d') be the edge of the star of A in $\dot{D}^A$ containing the vertex d, and so $d' \in D^A$. Put $P' = wPc, d, d'$. If P' is a passive A-path then by the definition of C^A, we have $d' \in C^A$, a contradiction. If P' is not a passive A-path, then by **4.5**, G has an active A-path, a contradiction. □

From **4.15**, **4.16**, and **4.18** we have

4.19. *Let A be an induced n-packing, and let $a \in T^A$. Suppose that G has no active A-trail. Then N^a is a connected component of $G\backslash H^A$, i.e.* $\mathrm{Cmp}(\dot{C}^A) \subseteq \mathrm{Cmp}(G\backslash H^A)$.

Since each vertex of H^A is adjacent to at least one vertex (actually at least two vertices) of C^A in G, we have from **4.18**

4.20. *Let A be an induced n-packing, and let $a \in T^A$. Suppose that G has no active A-trail. Then H^A is the set of vertices in $G\backslash C^A$ adjacent to at least one vertex in C^A.*

We now can describe the structure of a subgraph N^a.

4.21. *Let A be an induced n-packing, and let $a \in T^A$. Suppose that G has no active A-trail. Then N^a is an odd clique tree.*

Proof (Uses **4.14**). Let us prove the statement by induction on the number of vertices of N^a. If N^a has one vertex, i.e. $N^a = a$, then the statement is obviously true. So let $|V(N^a)| \geqslant 2$. Since $|V(N^a)|$ is odd, clearly $|V(N^a)| \geqslant 3$. By the definition of N^a, there exist $x_1, x_2 \in V(N^a)$ such that $\Delta = G[a, x_1, x_2]$ is a triangle, and the edge $e = (x_1, x_2)$ forms a small star of A. Let $\bar{G}$ and $\bar{N}^a$ be obtained from $G \backslash E(\Delta)$ and $N^a \backslash E(\Delta)$, respectively, by identifying x_1 and x_2 with the vertex a. Let $\bar{A} = A \backslash e$. It is easy to see that $\bar{A}$ is an induced n-packing in $\bar{G}$, $a \in T^{\bar{A}}(\bar{G})$, and $\bar{N}^a$ is an $\bar{A}_1$-subgraph of $\bar{G}$. It is also clear that every active $\bar{A}$-trail in $\bar{G}$ can be easily transformed into an active A-trail in G. Since G has no active A-trail, $\bar{G}$ also has no active $\bar{A}$-trail. Clearly $|V(\bar{N}^a)| = |V(N^a)| - 2$. Therefore by the inductive hypothesis, $\bar{N}^a$ is an odd clique tree.

Let $\bar{R}$ be a block of $\bar{N}^a$ that does not contain a. Then clearly $\bar{R}$ is also a block of N^a. Since $\bar{N}^a$ is an odd clique tree, $\bar{R}$ is an odd complete graph.

Let $\bar{C}$ be a block of $\bar{N}^a$ containing a. Then every vertex q of $\bar{C} \backslash a$ is adjacent to a vertex of Δ in G.

Suppose that there exists a vertex z of Δ such that $(z, q) \in E(G)$ and $(t, q) \notin E(G)$ for every $q \in V(\bar{C} \backslash a)$ and every $t \in V(\Delta) \backslash z$. Then $C = G[\bar{C} \backslash a \cup z]$ is an odd complete graph and is a block of N^a. In particular if $z = a$ then $C = \bar{C}$ is a block of N^a containing a. If $z \neq a$ then C is a block of N^a avoiding a. So we may conclude in particular that every block of N^a avoiding a is an odd complete graph.

Now suppose that there exist two distinct vertices z_1 and z_2 of Δ and two distinct vertices q_1 and q_2 of $\bar{C} \backslash a$ such that $(z_1, q_1) \in E(G)$ and $(z_2, q_2) \in E(G)$. Put $M(\bar{C}) = A \cap (\bar{C} \backslash a)$. Clearly $M(\bar{C})$ is a perfect matching of $\bar{C} \backslash a$. Therefore since $\bar{C} \backslash a$ is a complete graph and $\Delta = G[a, x_1, x_2]$ is a triangle containing a small star $G[x_1, x_2]$, the subgraph $F = G[\Delta \cup \bar{C} \backslash a)]$ has an A_1-cycle Q such that either $V(Q) = V(C)$ and $Q \cap T^A = a$, or $V(Q) = V(F \backslash a)$ and $(a, x_i) \in E(G)$. Since G has no active A-trail, by **4.14**, C is a complete graph and $M(C)$ is a perfect matching of $F \backslash a$.

Suppose that there are two different complete subgraphs F_1 and F_2 of G such that $\Delta \subset F_i$ and $M(F_i)$ is a perfect matching of $F_i \backslash a$, $i = 1, 2$. Then $H = F_1 \cup F_2$ contains an A_1-cycle L such that either $V(L) = V(H)$ and $L \cap T^A = a$, or $V(L) = V(H \backslash a)$ and $(a, x_i) \in E(G)$. Since G has no active A-trail, by **4.14**, H is a complete graph and $M(H)$ is a perfect matching of $H \backslash a$. Thus every block of N^a containing a is an odd complete graph. Since N^a is a connected graph, and every its block is an odd complete graph, N^a is an odd clique tree. □

Let Oct(F) denote the set of components of F that are odd clique trees. From **4.17** and **4.21** we have

4.22. *Let A be an induced n-packing of G. Suppose that G has no active A-trail. Then every component of $\dot{C}^A$ is an odd clique tree, i.e.* $\mathrm{Cmp}(\dot{C}^A) = \mathrm{Oct}(\dot{C}^A)$.

4.23. *Let T be an odd clique tree. Then T is an IS_n-critical graph for every integer $n \geqslant 1$ (and in particular, T is a matching-critical graph).*

Proof. (p1) Let us prove that $T' = T\backslash x$ has a perfect matching for every $x \in V(G)$. Clearly T' has the properties: (Pr1) every block of T' is a clique, and (Pr2) every component of T' has exactly one maximal clique with an even number of vertices.

Let us prove by induction of the number of vertices that a graph with properties (Pr1) and (Pr2) has a perfect matching. Clearly the statement holds for the graph with one edge. Let T'' be obtained from T' by deleting an arbitrary pair of (adjacent) vertices. Clearly T'' also has properties (Pr1) and (Pr2). By the inductive hypothesis, T'' has a perfect matching. Therefore T' also has a perfect matching.

(p2) Now let us prove by induction on the number of vertices that T has no perfect induced k-packing for every integer $k \geqslant 1$. Clearly the statement holds for the trivial graph with one vertex. Suppose that T has a perfect induced n-packing A for some integer $n \geqslant 1$. Let S be a star of A, and let $T' = T\backslash V(S)$. Then at least one component C of T' is an odd clique tree. By the inductive hypothesis, T' has no perfect induced n-packing for every integer $k \geqslant 1$. On the other hand, $A \cup C$ is a perfect induced n-packing of C, a contradiction. □

From **4.3**, **4.17**, **4.19**, **4.22**, and **4.23** we have

4.24. *Let A be an induced n-packing, and let $a \in T^A$. Suppose that G has no active A-trail. Then* $\mathrm{Oct}(G\backslash H^A) = \mathrm{Oct}(\dot{C}^A) = \mathrm{Cmp}(\dot{C}^A) = \{N^a\colon a \in T^A\} \subseteq \mathrm{Cmp}(G\backslash H^A)$.

5. Duality theorem for the induced *n*-star packings in a graph

Now we are ready to prove the Duality Theorem 2.3.

First we shall establish the corresponding lower bound on the number of vertices $|W[A]|$ in a graph G that are not covered by a given induced n-packing A.

Given a vertex subset Z of a graph F let $\mathrm{Oct}_Z(F)$ denote the set of all components C of F such that C is an odd clique tree and $V(C) \subseteq Z$. If in particular $Z = V(F)$ then $\mathrm{Oct}_Z(F) = \mathrm{Oct}(F)$ is the set of components of F that are odd clique trees.

5.1. *Let A be an induced n-packing of G, and X and Z be arbitrary vertex subsets of G. Then $|W[A] \cap Z| \geqslant |\mathrm{Oct}_Z(G\backslash X)| - n|X|$, and in particular if $Z = V(G)$, $|W[A]| \geqslant |\mathrm{Oct}(G\backslash X)| - n|X|$.*

Proof (Uses **4.23**). Let $O_Z(F)$ denote the union of all components that are members of $\mathrm{Oct}_Z(F)$. Let A' be the subset of edges of A that are not incident to X. Let Q be the set of vertices in $O_Z(G\backslash X)$ that are not covered by the edge set A'. By **4.23**, an odd clique tree does not have a perfect induced n-packing (and so its vertices cannot be covered by an induced n-packing of G). Therefore every component in $\mathrm{Oct}_Z(G\backslash X)$ has at least one vertex that is not covered by the edge set of A', i.e. $Q \cap C \neq \emptyset$ for every component (odd clique tree) $C \in \mathrm{Oct}_Z(G\backslash X)$. Therefore $|Q| \geqslant |\mathrm{Oct}_Z(G\backslash X)|$. Since every star in the induced n-packing A has at most n edges, it follows that at most $n|X|$ vertices in Q can be covered by the edge set $A\backslash A'$. Therefore at least $|Q| - n|X| \geqslant |\mathrm{Oct}_Z(G\backslash X)| - n|X|$ vertices remain uncovered by A. $\square$

Since $|T^A| = |W[A]| + n|H^A|$, we have from **4.24**

5.2. *Let A be an induced n-packing in G. Suppose that G has no active A-trail. Then* $|W[A]| = |\mathrm{Oct}(G\backslash H^A)| - n|H^A|$.

Now we can easily prove the Duality Theorem 2.3.

5.3. *Let $n \geqslant 2$. Then*

$$\min\{|W[B]|: B \in \mathscr{PIS}_n(G)\} = \max\{|\mathrm{Oct}(G\backslash X)| - n|X|: X \subseteq V(G)\}$$
$$= |\mathrm{Oct}(G\backslash H^A)| - n|H^A| = |W[A]|$$

for every V-maximal (maximum) induced n-packing A in G.

Proof (Uses **5.1** and **5.2**). By **5.1**, $|W[B]| \geqslant \max\{|\mathrm{Oct}(G\backslash X)| - n|X|: X \subseteq V(G)\}$ for every induced n-packing B of G. By **3.3**, an active A-trail is an augmenting A-trail. Let A be a V-maximal induced n-packing of G. Then G has no augmenting A-trail (see **4.1**), and therefore has no active A-trail. By **5.2**, $|W[A]| = |\mathrm{Oct}(G\backslash H^A)| - n|H^A|$. Therefore

$$\min\{|W[B]|: B \in \mathscr{PIS}_n(G)\} = |W[A]| = |\mathrm{Oct}(G\backslash H^A)| - n|H^A|$$
$$= \max\{|\mathrm{Oct}(G\backslash X)| - n|X|: X \subseteq V(G)\}. \quad \square$$

From **3.3**, **4.1**, **5.1**, and **5.2** we have the following strengthening of Theorem 2.9.

5.4. *An induced n-packing A of G is V-maximum (V-maximal) if and only if G has no active A-trail.*

6. Matroid generated by the induced n-star packings

Let $\mathscr{IS}_n(G)$ denote the family of vertex subsets of G that can be covered by an induced n-star packing of G. In this section we will show that $\mathscr{IS}_n(G)$ is the independence set of a matroid, and we will describe the structure of a circuit of this matroid.

As we have already mentioned before $\mathscr{IS}_n$ has the hereditary property

(h) $X \subseteq Y \in \mathscr{IS}_n \Rightarrow X \in \mathscr{IS}_n$.

Therefore $\mathscr{IS}_n$ is an *independence family of subsets of* $V(G)$. Let A be a V-maximal induced n-star packing of G, i.e. $V[A]$ is a maximal set (a *base*) in $\mathscr{IS}_n$.

We shall use the following notation similar to that in Section 4: B_z^A is the set of vertices in G that can be reached from z by a passive A-path, $\dot{B}_z^A$ is the subgraph of G induced by B_z^A, i.e. $\dot{B}_z^A = G[B_z^A]$, S_z^A is the set of big stars of A in $\dot{B}_z^A$, T_z^A is the set of tails of all big stars in $\dot{B}_z^A$ plus the vertex z, i.e. $T_z^A = \bigcup\{T(S)\colon S \in \mathscr{S}_z(A)\} \cup \{z\}$, H_z^A is the set of heads of big stars in $\dot{B}_z^A$, i.e. $H_z^A = \{h(S)\colon S \in \mathscr{S}_z(A)\}$, $C_z^A = B_z^A \backslash H_z^A$, $\mathscr{N}_z(A) = \{N^a\colon a \in T_z^A\}$. Clearly $|\mathscr{N}_z(A)| = |T_z^A| = n|H_z^A| + 1$.

The following statement on C_z^A is similar to **4.4** on C^A.

6.1. *Let A be an induced n-packing of G. Then C_z^A is the set of the last vertices of passive A-paths of G starting at z.*

The next statement is similar to **4.19**.

6.2. *Let A be a V-maximal induced n-packing of G and $z \in W[A]$. Then every N^a, $a \in T_z^A$, is a connected component of $G\backslash H_z^A$.*

Proof (Uses **4.19** and **6.1**). Put $G^z = G\backslash H_z^A$. Let $a \in T_z^A$. Obviously $N^a \subseteq G^z$. Suppose on the contrary that N^a is not a component of G^z. Then there exists an edge (x, y) of G such that $x \in V(N^a)$, and $y \in V(G^z)\backslash V(N^a)$. By **4.19**, $y \in H^A$. Therefore $y \in H^A \backslash H_z^A$. By the definitions of H^A and H_z^A, there exists a big star S of A such that $h(S) = y$ and $S \nsubseteq \dot{B}_z^A$. Since $x \in N^a$ and $a \in T_z^A$, clearly $x \in C_z^A$. By **6.1**, there exists a passive A-path P of G starting at z and terminating at x. Then $P' = Pxeye_t t$, where $e_t = (y, t)$ is an edge of S and t is a tail of S, is a passive A-path from z to t. Therefore $e_t \in E(\dot{B}_z^A)$ for every $t \in T(S)$, and so $S \subseteq \dot{B}_z^A$, a contradiction. □

6.3. *Let A be a V-maximal induced n-packing of G, and $z \in W[A]$. Then*

(s1) *there is no induced n-packing of G that covers C_z^A, and*

(s2) *for every $x \in C_z^A$ there exists an induced n-packing A_x such that $V[A_x] = V[A]\backslash x \cup z$.*

Proof (Uses **3.2**, **4.21**, **5.1**, **6.1**, and **6.2**). Let us prove (s1). Put $C_z^A = Z$ and $H_z^A = X$. Let $a \in T_z^A$. Obviously $V(N^a) \subseteq Z$. By **6.2**, N^a is a component of $G\backslash X$. By **4.21**, N^a is an odd clique tree. Hence $|\mathrm{Oct}_Z(G\backslash X)| \geqslant |\mathscr{N}_z(A)| = |T_z^A| = n|H_z^A| + 1 = n|X| + 1$. Therefore by **5.1**, $|W[A] \cap Z| \geqslant |\mathrm{Oct}_Z(G\backslash X)| - n|X| \geqslant 1$. Thus every induced n-packing in G does not cover at least one vertex in C_z^A.

Let us prove (s2). Let $x \in C_z^A$. Then by **6.1**, there exists a passive A-path zPx of G from z to x. Then by **3.2**, $A_x = A \triangle E(zPx)$ is the required induced n-packing. □

Now we can describe a circuit of the independence set $\mathscr{I}\mathscr{S}_n$.

6.4. *Let A be a maximal induced n-packing of G. Then*
(a1) C_z^A *is a circuit of the independence set* $\mathscr{I}\mathscr{S}_n$ *of subsets in* $V(G)$, *and*
(a2) C_z^A *is the unique circuit of* $\mathscr{I}\mathscr{S}_n$ *in the vertex set* $V[A] \cup z$.

Proof (Uses **6.3**). Let us prove (a1). By **6.3**(s1), $C_z^A \notin \mathscr{I}\mathscr{S}_n$. By **6.3**(s2), $X \in \mathscr{I}\mathscr{S}_n$ for every proper subset $X \subset C_z^A$. Therefore C_z^A is a circuit of $\mathscr{I}\mathscr{S}_n$.

Let us prove (a2). Suppose on the contrary that there exists a subset C of $V[A] \cup z$ that is a circuit of $\mathscr{I}\mathscr{S}_n$ distinct from C_z^A. Since a circuit is a minimal subset of $V(G)$ that is not a member of $\mathscr{I}\mathscr{S}_n$, we have $C_z^A \nsubseteq C$. Therefore, there exists $x \in C_z^A \backslash C$. Then $C \subseteq V[A] \backslash x \cup z$. By **6.3**(s2), there exists an induced n-packing A_x such that $V[A_x] = V[A] \backslash x \cup z$. Therefore the induced n-packing A_x covers C, and so $C \in \mathscr{I}\mathscr{S}_n$, a contradiction. □

It is easy to prove the following fact for matroids [7].

6.5. *Let $\mathscr{A}$ be a family of subsets of a finite set E having the hereditary property: $X \subseteq Y \in \mathscr{A} \Rightarrow X \in \mathscr{A}$. Suppose that for every maximal subset B in $\mathscr{A}$ and for every element $e \in E \backslash B$ the set $B \cup z$ contains the unique circuit of the family $\mathscr{A}$. Then $\mathscr{A}$ is the independence set of a matroid defined on E.*

Now we can prove Theorem 2.10(m1).

6.6. *The set $\mathscr{I}\mathscr{S}_n(G)$ of vertex subsets of G is the independence set of a matroid.*

Proof (Uses **6.4** and **6.5**). Let B be a maximal subset in the independence set $\mathscr{I}\mathscr{S}_n(G)$ of subsets in $V(G)$, and let $z \in V(G) \backslash B$. By **6.4**, $B \cup z$ contains the unique circuit of the independence set $\mathscr{I}\mathscr{S}_n(G)$. Therefore by **6.5**, $\mathscr{I}\mathscr{S}_n(G)$ is the independence set of a matroid defined on $V(G)$. □

Let $\mathscr{M}\mathscr{I}\mathscr{S}_n(G)$ denote the matroid of G with the independence set $\mathscr{I}\mathscr{S}_n(G)$, i.e. $\mathscr{M}\mathscr{I}\mathscr{S}_n(G) = (V(G), \mathscr{I}\mathscr{S}_n(G))$.

We need some notions and notation from the matroid theory [15]: M is a matroid with the ground set E, $C(M)$ is the set of cyclic elements of M (i.e. the elements belonging to at least one circuit of M), and $L^*(M)$ is the set of coloops (i.e. one element cocircuits) of M. Given a base B (i.e. a maximal independent set) of M and an element $e \in E \backslash B$, let $C(B, e)$ denote the circuit of M that is a subset of $B \cup e$.

It is known [15] and it is easy to prove that

6.7. *Let M be a matroid. Then*
(m1) $C(M) = \bigcup \{C(B, e)\colon e \in E \backslash B\}$,
(m2) $C(M)$ *is the set of elements that do not belong to at least one base of M, and*
(m3) $L^*(M) = E \backslash C(M)$.

6.8. *Let A be a maximal induced n-packing of G. Then*
(s1) *C^A is the set of cyclic elements of the matroid $\mathscr{MIS}_n(G)$, and*
(s2) *$V(G)\backslash C^A = H^A \cup D^A$ is the set of coloops of the matroid $\mathscr{MIS}_n(G)$.*

Proof (Uses **4.4**, **6.1**, **6.7**(m1), and **6.7**(m3)). Let $\mathscr{M} = \mathscr{MIS}_n(G)$. Let us prove (s1). Since A is a V-maximal induced n-packing of G, clearly $B = V[A]$ is a base of the matroid $\mathscr{M}$. By **6.7**(m1), $C(\mathscr{M}) = \bigcup\{C(B,z)\colon z \in W[A]\}$. By **6.1**, $C(B,z) = C^A_z$ is the set of the last vertices of passive A-paths starting from the vertex z. By **4.4**, C^A is the set of the last vertices of all passive A-paths. Therefore $C^A = \bigcup\{C^A_z\colon z \in W[A]\} = C(\mathscr{M})$. Now (s2) follows from (s1) and **6.7**(m3). □

7. Induced n-packing structure of a graph

Now we are ready to prove the Structure Theorem 2.6 for induced n-packings in a graph similar to the Gallai–Edmonds Structure Theorem for matchings.

Let A be an induced n-packing of G. We recall some notations: B^A is the set of vertices of G that can be reached from $W[A]$ by a passive A-path, $\dot{B}^A$ is the subgraph of G induced by B^A (i.e. $\dot{B}^A = G[B^A]$), $D^A = V(G)\backslash B^A$, $\dot{D}^A = G\backslash \dot{B}^A$, S^A is the set of big stars of A in $\dot{B}^A$, $T^A = \bigcup\{T(S)\colon S \in \mathscr{S}^A\} \cup W[A]$, H^A be the set of heads of big stars in $\dot{B}^A$ (i.e. $H^A = \{h(S)\colon S \in \mathscr{S}^A\}$), and $C^A = B^A\backslash H^A$, and $\dot{C}^A = \dot{B}^A\backslash H^A$.

We also recall that $C_n(G)$ is the set of all vertices of G that are not covered by at least one V-maximum induced n-packing of G, $H_n(G)$ is the set of all vertices in $V(G)\backslash C_n(G)$ adjacent to at least one vertex in $C_n(G)$, and $D_n(G) = V(G)\backslash(H_n(G) \cup C_n(G))$. As above $\dot{C}_n(G)$ and $\dot{D}_n(G)$ are subgraphs of G induced by the vertex sets $C_n(G)$ and $D_n(G)$, respectively.

7.1. *Let G be an arbitrary graph, and n be an integer at least 2. Let A be a V-maximum (or V-maximal) induced n-packing of G. Then (see Fig. 1)*
(a1) *the components of the subgraph $\dot{C}_n(G)$ of G are* IS$_n$*-critical (are odd clique trees),*
(a2) *the subgraph $\dot{D}_n(G)$ has a perfect n-packing,*
(a3) *$C^A = C_n(G)$, $H^A = H_n(G)$, $D^A = D_n(G)$,*
(a4) *A contains*
(a4.1) *a near perfect matching of each component of the subgraph $\dot{C}_n(G)$,*
(a4.2) *a perfect n-packing of the subgraph $\dot{D}_n(G)$, and*
(a4.3) *a set of $|H_n(G)|$ disjoint big stars such that their heads are in $H_n(G)$, their tails are in $\dot{C}_n(G)$, and each component of $\dot{C}_n(G)$ contains at most one tail of all these big stars,*
(a5) $|W[A]| = \min\{|W[B]|\colon B \in \mathscr{PIS}_n(G)\} = |\mathrm{Oct}(G\backslash H_n(G))| - n|H_n(G)|$.

Proof (Uses **4.3**, **4.15**, **4.17**, **4.20**, **5.3**, **6.3**(s1), and **6.7**(m2)). By **6.3**(s1), C^A is the set of cyclic elements of the matroid $\mathscr{MIS}_n(G)$. By **6.7**(m2), C^A is the set of vertices of G that do not belong to at least one base of $\mathscr{MIS}_n(G)$. Since a base of $\mathscr{MIS}_n(G)$ is the vertex set covered by a V-maximal induced n-packing, we have $C^A = C_n(G)$. Therefore by **4.20**, $H^A = H_n(G)$, and so $D^A = D_n(G)$. Hence (a3) holds. Now (a4.1) follows from **4.15**(a2), (a4.2) follows from **4.3**, (a4.3) follows from (a3) and **4.17**, and (a5) follows from **5.3**. $\square$

Theorem 2.10(m2), (m3) follows from **6.4** and **7.1**(a3).

7.2. *The following conditions are equivalent*:

(c1) *F is an induced* IS_n*-critical graph, $n \geqslant 2$,*

(c2) *F is an induced* IS*-critical graph, and*

(c3) *F is an odd clique tree.*

Proof (Uses **4.23** and **7.1**). By **4.23**, an odd clique tree is an induced IS_n-critical graph, $n \geqslant 2$, and an induced IS-critical graph, i.e. (c3) $\Rightarrow$ (c1) and (c3) $\Rightarrow$ (c2). Clearly (c2) $\Rightarrow$ (c1). Let us prove (c1) $\Rightarrow$ (c3). Suppose that $V(F) \backslash C_n(F) \neq \emptyset$. Then by **7.1**, $F \backslash x$ does not have a perfect induced n-packing, and so F is not an induced IS_n-critical graph, a contradiction. Therefore $V(F) = C_n(F)$. By **7.1**, every component of $F = \dot{C}_n(F)$ is an odd clique tree. Since every induced IS_n-critical graph is connected, it follows that F is an odd clique tree. $\square$

Theorem 2.1 follows directly from **5.3** and **7.2**.

Now we can investigate the set of $\mathscr{IS}$-obstacles of G (see Theorem 2.8).

Let $\mathscr{F}$ be a set of graphs. Given a graph G and $X \subseteq V(G)$, let $f_G(X)$ denote the number of components of $G \backslash X$ that are members of $\mathscr{F}$. It is easy to see that

7.3. *$f_G(X)$ is a supermodular function of X, i.e. $f_G(X) + f_G(Y) \leqslant f_G(X \cup Y) + f_G(X \cap Y)$ for every two vertex subsets X and Y of G.*

Put $d_G(X) = \mathrm{Oct}(G \backslash X) - n|X|$ and $m = \min\{|W[A]|\colon A \in \mathscr{PIS}_n(G)\}$. We recall that $X \subseteq V(G)$ is an $\mathscr{IS}$-obstacle if $d_G(X) = m$.

7.4. *Let X and Y be $\mathscr{IS}$-obstacles in G. Then $X \cup Y$ and $X \cap Y$ are also $\mathscr{IS}$-obstacles in G.*

Proof (Uses **5.3** and **7.3**). By **7.3**, $\mathrm{Oct}(G \backslash X)$ is a supermodular function of X. Since $|X| + |Y| = |X \cup Y| + |X \cap Y|$, the function $d_G(X)$ is also supermodular, i.e. $d_G(X) + d_G(Y) \leqslant d_G(X \cup Y) + d_G(X \cap Y)$. Since X and Y are $\mathscr{IS}$-obstacles in G, we have $d_G(X) = d_G(Y) = m$. By **5.3**, $m = \max\{d_G(Z)\colon Z \subseteq V(G)\}$. Therefore $d_G(X \cup Y) \leqslant m$ and $d_G(X \cap Y) \leqslant m$. Hence $2m = d_G(X) + d_G(Y) \leqslant d_G(X \cup Y) + d_G(X \cap Y) \leqslant 2m$. From the above inequalities we have: $d_G(X \cup Y) = m$ and $d_G(X \cap Y) = m$, and so both $X \cup Y$ and $X \cap Y$ are $\mathscr{IS}$-obstacles in G. $\square$

Let $\mathscr{X}_n(G)$ denote the set of all $\mathscr{I}\mathscr{S}_n$-obstacles in G.

7.5. *Let $\mathscr{L}_n(G) = (\mathscr{X}_n(G), \subseteq)$ denote the set $\mathscr{X}_n(G)$ partially ordered by the inclusion operation $\subseteq$. Then*

(a1) *$\mathscr{L}_n(G)$ is a sublattice of the lattice of all subsets of $V(G)$ under inclusion,*

(a2) *$H_n(G)$ is the minimum element of $\mathscr{L}_n(G)$, and*

(a3) *if $X \in \mathscr{L}_n(G)$ then $X \backslash H_n(G) \in \mathscr{L}_n(\dot{D}_n(G))$.*

Proof (Uses **7.1** and **7.4**). Clearly (a1) follows from **7.4**.

Let us prove (a2). Put $H = H_n(G)$ and $W = W[A]$. Let A be a V-maximum induced n-packing in G. By **7.1**, $H = H^A$ and $\dot{C}_n(G) = \dot{C}^A$. By **4.17**, $\mathrm{Cmp}(\dot{C}^A) = \{N_a \colon a \in T^A\}$.

Let $Z \subseteq T^A$ and $Y \subseteq H$. Put $\mathscr{N}(Z) = \{N^a \colon a \in Z\}$, $\mathscr{N}^h = \{N^a \colon a \in T(S^h\}$ for $h \in H$, and $\mathscr{N}^Y = \bigcup\{\mathscr{N}^h \colon h \in Y\}$. Obviously $|\mathscr{N}(Z)| = |Z|$. Since $h \in H$, clearly $|T(S^h| = n$, and so $|\mathscr{N}^h| = |T(S^h| = n$. Therefore $|\mathscr{N}^Y| = n|Y|$. Clearly $\mathrm{Oct}(G\backslash Y) \subseteq \mathscr{N}(W) \cup \mathscr{N}^Y$, and so $|\mathrm{Oct}(G\backslash Y)| \leqslant |\mathscr{N}(W)| + |\mathscr{N}^Y| = |W| + n|Y|$.

Let $X \subset H$. It is sufficient to prove that X is not an $\mathscr{I}\mathscr{S}_n$-obstacle in G, i.e. that $|\mathrm{Oct}(G\backslash X)| - n|X| < |W|$. Suppose on the contrary that $|\mathrm{Oct}(G\backslash X) - n|X| \geqslant |W|$. Since $|\mathrm{Oct}(G\backslash X)| - n|X| \leqslant |W|$, we have $|\mathrm{Oct}(G\backslash X)| - n|X| = |W|$. Then $\mathrm{Oct}(G\backslash X) = \mathscr{N}(W) \cup \mathscr{N}^X$. Put $O^X = \bigcup\{K \colon K \in \mathrm{Oct}(G\backslash X)\}$. Let $h \in H\backslash X$. Then $h \notin O^X$ and h is adjacent to no vertex in O^X. Since h is a vertex of B^A, there exists a passive A-path wPz containing h, so that $h \in P$ and $w \in W$. By the above equation, $w \in W \subseteq O^X$. Therefore wPh has an edge connecting h with a vertex v in $O^X \cup X$. Since h is adjacent to no vertex in O^X, clearly $v \in X$. Since $(h, v) \in E(G)\backslash A$, the subpath wPv of P is a passive A-path, and so the last edge (v', v) of wPv is an edge of the big star S^v with the head v and a tail v'. Therefore the tail v' preceeds the head v of S^v in wPv. Hence wPv is not a passive A-path, a contradiction.

Now (a3) follows from (a1) and (a2). □

8. Polynomial algorithm for packing induced stars in a graph

Let A be an induced n-packing of G. Let F be a subgraph of G.

We use the following notation similar to that in Section 4: $S^A(F)$ is the set of big stars of A in F, $T^A(F) = \bigcup\{T(S) \colon S \in S^A(F)\} \cup (W[A] \cap F)$, $H^A(F) = \{h(S) \colon S \in S^A(F)\}$, and $F^A = F\backslash H^A(F)$.

A subgraph F of G is called an *A-subgraph* if

(a1) $W[A] \subseteq V[F]$,

(a2) if S is a star of A in F then S is either a big star or a small star,

(a3) if S is a star of A and $S \cap F \neq \emptyset$ then $S \subseteq F$,

(a4) every component of F contains exactly one vertex from $W[A]$,

(a5) every component of F^A is an odd clique tree,

(a6) every component of F^A contains exactly one vertex from $T^A(F)$,

(a7) if K is a component of F^A then $K\backslash T^A(F)$ has a perfect matching $M(K)$ consisting of small stars of A (i.e. $M(K) \subseteq A_1$).

Let us now give an (informal) description of an algorithm that for a given graph G finds a vertex maximum induced n-packing A as well as $H_n(G)$. A similar algorithm can be used to find for $X \subseteq V(G)$ an induced n-star packing that covers X (if any).

The general step deals with the current information $(A, F, H, \mathscr{P})$ where A is an induced n-packing A, F is an A-subgraph of G, $H = H^A(F)$, and $\mathscr{P} = \{P_x\colon x \in V(F^A)\}$ where P_x is a passive A-path in F terminating at x.

In the first step $A := \emptyset$ (and so $W[A] = V(G)$), and F is the subgraph of G with $V(F) = W[A] = V(G)$ and $E(F) = \emptyset$, and so $H = H^A(F) = \emptyset$ and $P_x = x$ for $x \in V(G)$.

In every step we have one of the following goals:

(g1) enlarge F, i.e. find an A-subgraph F' of G such that $F \subset F'$, and

(g2) enlarge A by finding an active A-trail Q and putting $A' = A \triangle Q$, so that $V(A) \subset V(A')$.

If we cannot reach either of these goals then this step is final, the current induced n-packing A of this step is V-maximum.

Now we shall describe a general step of the algorithm.

General step S. Find (if any) an edge $e = (x, y)$ of G having the following property **(p)** $x \in F^A$ and $(x, y) \in E(G) \backslash E(F)$.

If there is no such edge then stop: A is V-maximum and $H = H^A(F) = H_n(G)$.

So suppose we have found an edge $e = (x, y)$ with the property **(p)**. Let $P_x = wPx$. There are three options for y:

(c1) $y \in H^A(F)$,

(c2) $y \in V(F) \backslash H^A(F)$, and

(c3) $y \in V(G) \backslash V(F)$.

In case (c1), put $F' = F \cup e$, $H' = H$, and $\mathscr{P}' = \mathscr{P}$.

Let us consider case (c2). Suppose that x and y belong to different components of F^A. Then find an active A-trail Q. [By **4.16**, an active A-trail exists. The proof of **4.16** shows how to find it by using the passive A-path wPx].

Now suppose that x and y belong to the same component K of F^A. Since F is an A-subgraph of G, the component K is an odd clique tree. Hence we can find a (unique) chain $L = (x = b_1)B_1 b_2 \cdots B_s(b_{s+1} = y)$ where B_i is a block of K and $b_i, b_{i+1} \in V(B_i)$, $i = 1, \ldots, s$. If $G[L]$ is a complete graph, put $F' = F \cup G[L]$, $H' = H$, and $\mathscr{P}' = \mathscr{P}$. If $G[L]$ is not a complete graph then find an active A-trail Q. [By **4.14**, an active A-trail exists, and the proof of **4.14** shows how to find it by using the passive A-path wPx.]

In case (c3), find the star S^y containing y [clearly such star exists, and $F \cap S^y = \emptyset$]. S^y can be either a small star or an intermediate star or a big star. If S^y is a non-small star then y can be either a tail or the head of S^y.

Suppose that S^y is a small star of A, say $G[y, z]$. Then check whether (x, z) is an edge of G. If $(x, z) \in E(G)$ then put $F' = F \cup G[x, y, z]$, $H' = H$, and $\mathscr{P}' = \mathscr{P} \cup \{P_y, P_z\}$ where $P_y = wPx, z, y$ and $P_z = wPx, y, z$. If $(x, z) \notin E(G)$ then $Q = wPx, y$ is a 3-active A-path.

Now suppose that S^y is a non-small star. Let $h = h(S^y)$.

Suppose that y is a tail of S^y. Then $Q = wPx, y, h$ is a 4-active A-path.

Suppose that y is the head of S^y: $y = h$. Then find a tail t of S^y adjacent to x.

Suppose that such tail exists. Then $Q = wPx, t, h$ is a 4-active A-path.

Now suppose that no tail is adjacent to x. If S^y is an intermediate star, then $Q = wPxey$ is an 2-active A-path. If S^y is a big star, then put $F' = F \cup S^y \cup e$, $H' = H \cup y$, and $\mathscr{P}' = \mathscr{P} \cup \{P_t\colon t \in T(S^y)\}$ where $P_t = wPx, y, t$.

Thus we have one of the following two outcomes:

(d1) a new A-subgraph F' such that $F \subset F'$, and

(d2) an active A-trail Q.

In case (d1), put $A' = A$. In case (d2), let $A' = A \triangle E(Q)$ and F' be the subgraph with $V(F') = W[A']$, $E(F') = \emptyset$, $H' = \emptyset$, and $\mathscr{P}' = \{P_x\colon x \in V(F')\}$ where $P_x = x$.

Now repeat step **S** with the new information $(A, F, H, \mathscr{P}) := (A', F', H', \mathscr{P}')$.

Clearly the algorithm described above is polynomial although far from being the best. For example, in case (d2) a 'better' F' can be found by an appropriate modification of F. Also it is sometimes easier to find an augmenting A-trail instead of active A-quasi-path.

This algorithm can be used to give alternative proofs of the main results.

A more efficient algorithm and an alternative proofs of the main results based on this algorithm will be given in another paper.

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